ON A METHOD OF SIMPLIFICATION OF SWITCHING FUNCTIONS FOR SYNTHESIS OF THREE LEVEL CIRCUITS

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(*Received November* 11, 1963)

ABSTRACT. A method of simplification of switching functions for synthesis of three-level circuits is presented. The method is an extension of the technique for simplification of switching functions for realising two-level circuits. The simplified expressions for synthesis of three-level circuits are obtained from the irredundant two-level covers of the functions by properly grouping the basic cells in the irredundant covers.

INTRODUCTION

A switching function $f(x_{n-1}, x_{n-2}, \ldots, x_0)$ of n binary variables x_{n-1}, x_{n-2}, ..., x_0 can be written either as a sum of product terms or as a product of sum terms of the variables, the former being called the disjunctive normal form and the latter the conjunctive normal form of switching functions. Thus, when written as a sum of products, the switching function is expressed as

$$f(x_{n-1}, x_{n-2}, \ldots, x_0) = \Sigma\, g_i \tag{1}$$

where g_i's are each a product term of n or fewer number of variables. In this form, the switching function can be mechanised by a set of AND circuits feeding into an OR circuit. Alternatively, when the switching function is expressed as the product of the sum terms, the function is written as

$$f(x_{n-1}, x_{n-2}, \ldots, x_0) = \Pi q_i \tag{2}$$

where q_i's are the sum terms. This functional form is mechanised by a set of OR circuits feeding into an AND circuit. In the limiting case when the function can be expressed by a single product or sum term, a single AND or OR circuit is required for the mechanisation of the function. For certain functions, a two-level AND–OR realisation might give more economy than its OR–AND realisation in respect of the number of logical circuit elements (diodes or relay contacts). For others the reverse may be true. Methods have been suggested by several authors for obtaining the minimal two-level switching circuit in either of the two forms described above (Quine, 1952; 1955; Veitch, 1952; Karnaugh, 1953; Mc-

Cluskey, 1956; Urbano and Mueller, 1956; Roth, 1957; Svoboda, 1957; Mukhopadhyay, 1961; Choudhury and Basu, 1962).

It is well-known that many switching functions when mechanised by three-level OR-AND-OR or AND-OR-AND circuits require lesser number of logical circuit elements. For three-level circuit realisation a switching function is written as

$$f(x_{n-1}, x_{n-2}, \ldots, x_0) = q_1 q_2 \cdots + q_{1a} q_{2a} \cdots + \cdots \qquad \ldots \quad (3)$$

or as
$$f(x_{n-1}, x_{n-2}, \ldots, x_0) = (g_1+g_2+\ldots)(g_{1a}+g_{2a}+\ldots)\ldots \qquad (4)$$

where each of the terms of the form $q_1 q_2 \ldots$ or $(g_1+g_2+\ldots)$ is the product of the sum terms or the sum of the product terms respectively and can be realised by two-level circuits, the overall function being mechanised by three-level circuits. In the form of representation given in expression (3), the function is realised by an OR-AND-OR circuit whereas the form given in expression (4) can be mechanised by an AND-OR-AND circuit. The simplified three-level OR-AND-OR form of a switching function can be arrived at by splitting the function into a number of component functions and realising each of the component functions in OR-AND configuration. The simplified AND-OR-AND circuit realisation of the switching function can be obtained by first finding out the OR-AND-OR expression of the complementary function and then complementing the complementary function.

The available methods (Abhyankar, 1958; Veitch, 1959; Hurley, 1961) of realising OR-AND-OR or AND-OR-AND circuits of a switching function can be classified as

(i) inhibiting method (Hurley, 1961),

or (ii) pattern recognition in Veitch-Karnaugh maps (Veitch, 1959).

These methods though simple and straightforward suffer from the disadvantage that for their application one should be thoroughly acquainted with the functional patterns on the Veitch-Karnaugh map. Further, the difficulty of the pattern recognition in Veitch-Karnaugh map increases with the increase in the number of variables. In the present paper we shall suggest a method of obtaining simplified three-level expression of a switching function as a direct extension of the two-level minimisation process. The method mainly consists in finding the sets of basic k-cells of the function that are incident in $(k+1)$-cells. Only the basic cells in the irredundant covers of the function have to be searched for such incidence. A number of theorems has been proved in this connection, by the application of which three-level forms of a switching function are readily found out.

n-Dimensional Cube as a Model of Switching Functions :

With n binary variables, 2^n different terms can be obtained where each term is a product of all the variables, either primed or unprimed. The terms can also be represented as vertices of an n-cube. Such an n-cube is made up of cells of different dimensions (Urbano and Mueller, 1956; Svoboda, 1957), e.g.,

0-cell or vertex	a point	$k = 0$
1-cell	a line	$k = 1$
2-cell	a quadrilateral	$k = 2$
3-cell	a cube	$k = 3$
k-cell		$k = k$

where k denotes the dimension of the cells. Total number of k-cells present in an n-cube is ${}^nC_k \cdot 2^{n-k}$ where k may have any value from 0 to n. A switching function can be thought to be a collection of a set of vertices taken out of the 2^n vertices of the n-cube. The cell complex and the basic cells of a given switching function will obviously depend on the vertices of the n-cube chosen for the representation of the function. If the form representing a switching function be an irredundant cover then the cell complex of the function consists of basic cells only. A minimal cover which consists of basic cells only is also an irredundant cover.

If any two vertices in the n-cube differ in k variables in their algebraic representations then evidently they agree in $(n-k)$ variables so that $(n-k)$ variables are common in the algebraic representations of these two vertices. Since a k-cell requires $(n-k)$ variables in its representation, there can be one and only one k-cell in the n-cube which can include these two vertices jointly. In such situations we will say that the vertices are incident in that particular k-cell. If in a function there are two basic k-cells, such that there are $n-(k+1)$ variables common in their algebraic representation, there can be one and only one $(k+1)$-cell in the n-cube which includes these two k-cells jointly. These two k-cells are then similarly said to be incident in that particular $(k+1)$-cell.

Likewise if b basic k-cells of the n-cube are incident in a common $(k+1)$-cell, then in the disjunctive normal form of the function there will be disjunction of b terms where each term is the conjunction of $(n-k)$ variables and these terms will have $n-(k+1)$ variables common between them. The equivalent conjunctive normal form covering these basic k-cells will be the conjunction of two terms, one of which is the disjunction of b variables and the other conjunction of n-$(k+1)$ variables.

Basic Concepts and Theorems in Connection with Three-Level Simplification :

Let us consider a cover of a switching function of n variables consisting only of b basic k-cells and let these basic k-cells be incident in a common $(k+1)$-cell. Then all these basic k-cells will be essential cells because if any of these basic k-cells is removed, then some of the vertices of the function will remain uncovered. The disjunction of these b terms representing b, k-cells is the minimal form of the function. If we complement the function, multiply out the terms and drop all the subsuming terms we will get the basic cells of the complementary function. In this complementary function also, all the basic cells will be essential cells since if any of these basic cells of the complementary function is dropped then the disjunction of the remaining basic cells will not be a cover of the complementary function.

Hence if b basic k-cells representing a switching function are incident in a common $(k+1)$-cell, then the minimal [AND-OR and OR-AND] expressions of the function can be written either as

$$f(x_{n-1}, x_{n-2}, \ldots, x_0) = g_1+g_2+\ldots+g_b = \text{Disjunction of } b \text{ terms where each term is the conjunction of } (n-k) \text{ variables} \ldots \quad (5)$$

or as $$f(x_{n-1}, x_{n-2}, \ldots, x_0) = g_k \cdot q_l = [\text{Conjunction of } n-(k+1) \text{ variables}].[\text{Disjunction of } b \text{ variables}] \quad \ldots \quad (6)$$

where the variables of g_k and q_l are all mutually exclusive.

Therefore if any switching function consists of b basic k-cells and if these k-cells are incident in a common $(k+1)$-cell, then the function may be mechanised by two different forms of two-level circuits. In one of the forms, we realise each of the b k-cells separately so that the overall function can be mechanised by b AND circuits each with $(n-k)$ inputs feeding into an OR circuit with b inputs. Alternately, the function can be mechanised by an OR circuit with b inputs feeding into an AND circuit with $(n-k)$ inputs. This second form of representation is evident from expression (6).

Total number of diodes required for the mechanisation of the function given in expression (5) i.e., as sum of b k-cells is $C_1 = [b(n-k)+b]$ whereas that required for the mechanisation of the function from expression (6) i.e., by an OR-AND circuit is $C_2 = (n-k+b)$. Table I gives the relative values of C_1 and C_2 for different values of n, k and b.

TABLE I

n	k	b	C_1	C_2
4	1	2	8	5
5	1	2	10	6
6	1	2	12	7
7	1	2	14	8
8	1	2	16	9
4	2	2	6	4
5	2	2	8	5
6	2	2	10	6
7	2	2	12	7
8	2	2	14	8
4	2	3	9	5
5	2	3	12	6
6	2	3	15	7
7	2	3	18	8
8	2	3	21	9

A formal inspection of this table shows that economy can always be effected if a function be mechanised by an OR-AND circuit when the cell complex of the function is such that there are b basic k-cells incident in a $(k+1)$-cell.

If in an irredundant two-level cover of a switching function there are a number of different basic cells of which b basic k-cells are incident in a $(k+1)$-cell so that these k-cells can be realised in the OR-AND form, then the overall functional representation becomes a third-order expression and the function can be mechanised by a three-level OR-AND-OR circuit. To facilitate three-level realisations of switching function starting from its two-level irredundant covers, we shall discuss certain properties of the associated basic cells in the irredundant covers.

Theorem 1 : *The maximum number of basic k-cells that can be incident in a $(k+1)$-cell without covering all the vertices of the $(k+1)$-cell is $(k+1)$.*

Proof : When a number of basic k-cells are incident in a common $(k+1)$-cell, they can be represented as the product of two terms $q_k q_l$ as shown in expression (6), where the variables of the terms are mutually exclusive. Of these two terms, g_k is a conjunction of $n-(k+1)$ variables. So the disjunction terms q_l can consist of a maximum of $(k+1)$ variables. Therefore the maximum number of basic k-cells that can be incident in a common $(k+1)$-cell without covering the $(k+1)$-cell is $(k+1)$.

Theorem 2 : *If b basic k-cells of a given function are incident in a common $(k+1)$-cell, the total number of different vertices covered by b basic k-cells is*

$$v_b = 2^k + 2^{k-1} + \ldots + 2^{k-b+1} \qquad \ldots \quad (7)$$

Proof : If b basic k-cells are incident in a common $(k+1)$-cell they can be represented by the product of two terms $g_k q_l$ as shown in expression (6) where g_k is the n-$(k+1)$ variable term and q_l is the disjunction term $(x_1+x_2+\ldots+x_b)$ where the variables may be primed or unprimed. In this $g_k x_1$ will cover 2^k terms of the canonical form of the given function, corresponding to the vertices covered by the k-cell $g_k x_1$. Each such term covered by the k-cell $g_k x_1$ is represented by the conjunction of n variables, either primed or unprimed. Of these n variables, $(n-k)$ variables that are used for the representation of the k-cell $g_k x_1$ are common for all the terms covered by the cell. Rest of the variables (i.e., k variables) not included in the representation of the k-cell are present in their all possible combinations. Such is the case for the k-cell $g_k x_2$. Therefore of all the terms covered by the two k-cells, terms involving the variables g_k, x_1 and x_2 are common between the two k-cells. The number of such common terms is obviously

$$\frac{2^k}{2} \text{ or } 2^{k-1}$$

Therefore the total number of different terms or vertices covered by the two k-cells incident in a common $(k+1)$-cell but not covering a $(k+1)$-cell is

$$v_2 = 2^k + 2^{k-1}$$

If there are b basic k-cells incident in a $(k+1)$-cell, by similar argument it can be shown that the total number of different vertices covered by those b basic k-cells is

$$v_b = 2^k + 2^{k-1} + \ldots + 2^{k-b+1}$$

The above result gives us a very easy procedure of testing whether a certain number of basic k-cells of a given switching function are incident in a common $(k+1)$—cell or not.

Example : Show whether the following two-cells are incident in a three-cell :

Case (*i*). Basic 2-cells of the function are :

(1) $(0, 1, 2, 3) = A$

(2) $(0, 1, 4, 5) = B$

(3) $(0, 1, 8, 9) = C$

The total number of different terms covered by the three two-cells A, B, C is $v_b = 8$; but for three two-cells to form part of a three-cell the total number of different terms to be covered should be $v_b = 7$. Therefore the three two-cells cannot be incident in a three-cell. But 2-cells (A, B) or (B, C) or (A, C) are incident in different 3-cells.

Case (ii). Basic two-cells of the function are :

(1) $(0, 1, 2, 3) = D$

(2) $(0, 1, 4, 5) = E$

(3) $(0, 2, 4, 6) = F$

The total number of different terms covered by the three two-cells (D, E, F) is $v_b = 7$. Therefore these three two-cells are incident in a three cell.

We have already mentioned that if in an irredundant cover of a switching function, certain number of basic k-cells be incident in a common $(k+1)$-cell then those k-cells can be realised by OR-AND configuration with the net result that the overall function can be realised in three-level form and by such realisation economy may be achieved as compared to the two-level minimal form. The justification of starting from basic cells present in the irredundant covers as embracing a group of terms for three-level realisation lies in the fact that the total number of logical circuit elements (i.e., diodes) required to realise a basic $(k+1)$-cell as the sum of two or more k-cells is always greater than that required to realise the same as a $(k+1)$-cell, that is, as the conjunction of n-$(k+1)$ variables.

We shall next show that by introducing redundant basic cells in the irredundant covers of switching functions, the resulting three-level realisations that we may obtain, will give under no circumstances economy as compared to the two-level realisations.

Theorem 3 : *If in an irredundant cover of a switching function consisting of a number of basic k-cells not incident in $(k+1)$-cell, a k-cell is introduced from the subset of basic cells not included in the irredundant cover of the function, then by such incorporation of a redundant k-cell in the irredundant cover for three-level realisation, no economy is achieved.*

Proof : Since the k-cells present in the irredundant cover do not form part of a $(k+1)$-cell, therefore the number of common variables in the terms representing the k-cells should be less than $(n-k-1)$. If we now introduce a k-cell which can form part of a $(k+1)$-cell together with a k-cell already present in the irredundant cover, then $(n-k-1)$ variables must be common between the two. And these $(n-k-1)$ variables cannot be in common with any other k-cell. That is, by introducing one k-cell in the irredundant cover from the subsets of basic cells not included in the irredundant cover, we realise only one k-cell of the irredundant cover as being incident in a $(k+1)$-cell. Further, the total number of diodes required for realisation of a k-cell is equal to $(n-k)$. The total number of diodes required to realise two k-cells which are incident in a $(k+1)$-cell [as given in expression (6)], is $(n+2-k)$ and is always greater than $(n-k)$. Therefore it follows that no economy can be effected in three-level realisation by incorporating a redundant k-cell in the irredundant cover.

If in an irredundant cover of a switching function, two or more of the basic k-cells are not incident in a $(k+1)$-cell, then ordinarily, under the conditions we are considering, we do not get any three-level cover. But certain operations and manipulation with the basic cells of this irredundant cover might lead to 3-level circuits with better economy over the irredundant two-level cover. The following theorem gives conditions under which such economic realisation might be possible.

Theorem 4 : *If in an irredundant cover of a switching function, the points that are uncovered due to the removal of a basic $(k+1)$-cell are covered by a k-cell belonging to the body of that basic $(k+1)$-cell and if this k-cell along with another basic k-cell in the irredundant cover be incident in a $(k+1)$-cell, then replacement of the basic $(k+1)$-cell by the k-cell which forms its part will lead to greater economy if the function is realised by a three-level circuit.*

Proof : Total number of diodes required for the realisation of the basic $(k+1)$-cell and the k-cell is

$$[n-(k+1)]+(n-k)+2 = n-k+1+n-k = 2(n-k)+1$$

Total number of diodes required for the realisation of the two k-cells incident in a $(k+1)$-cell to replace these basic $(k+1)$-cell and k-cell is $(n-k+2)$. Since in the case that we are considering $(n-k)$ will always be greater than one, the sum $(n-k+2)$ will be always less than $2(n-k)+1$. It follows that under these conditions a three-level form will be more economical than the two-level irredundant form. If the irredundant cover under consideration happens to be a minimal cover, then obviously the three-level circuit will require lesser number of circuit elements than the two-level circuit.

Theorem 5 : *In an irredundant cover, if a k- cell which is not included in a given $(k+1)$-cell can, along with k-cells of that $(k+1)$-cell be incident in different $(k+1)$-cells, then there can be only two different k-cells in the given $(k+1)$-cell with which it can so combine for incidence in two different $(k+1)$-cells.*

Proof: Total number of variables required to represent a $(k+1)$-cell is $n-(k+1)$ whereas that required to represent a k-cell is $(n-k)$. If this k-cell along with a k-cell belonging to the body of the $(k+1)$-cell can be incident in a different $(k+1)$-cell, then these two k-cells can be represented as

$$g_k(x_i+x_j)$$

where the term g_k is the conjunction of $n-(k+1)$ variables and x_i and x_j are single variable terms, all the variables being mutually exclusive. If such a representation is possible then it is evident that $(n-k-2)$ variables must be common in the expression representing the $(k+1)$-cell and k-cell. Therefore it follows that in the body of the $(k+1)$-cell, there can be only two k-cells which can be incident in two different $(k+1)$-cells being combined with the existing k-cell.

As a consequence of theorems 4 and 5 it follows that if a k-cell belonging to the body of a basic $(k+1)$-cell be incident in a different $(k+1)$-cell being combined with the k-cells present in the irredundant cover, the total number of individual points or terms remaining uncovered by the withdrawal of the basic $(k+1)$-cell should not be greater than $2^k/2$ i.e., 2^{k-1}. This evidently gives us a very easy method of recognising whether we can eliminate a $(k+1)$-cell and a k-cell and replace them by two k-cells which can be incident in a $(k+1)$-cell for economic three-level realisation. If the points remaining uncovered be greater than 2^{k-1} then under no circumstances we can replace a basic $(k+1)$-cell in the irredundant cover by k-cells so as to be incident in different $(k+1)$-cells being combined with the existing basic k-cells. But if the points remaining uncovered due to such removal be less than 2^{k-1}, then we may try to select k-cells from the body of that basic $(k+1)$-cell, such that these k-cells cover points remaining uncovered due to removal of the $(k+1)$-cell, and combine with the k-cells already present in the irredundant covers so as to be incident in different $(k+1)$-cells.

PROCEDURE FOR OBTAINING THREE-LEVEL COVERS

It is evident from the theorems proved earlier, that three-level realisations of switching functions can be obtained from its two-level irredundant covers. Detailed procedure for three-level simplification is given below :

(1) The prime implicants i.e., basic cells and the irredundant covers of the given switching function are first found out by any standard method (Quine, 1952; 1955; Veitch, 1952; Karnaugh, 1953; McCluskey, 1956; Urbano and

Mueller, 1956; Petrick, 1956; 1959;, Roth, 1957; Svoboda, 1957; Ghazala, 1957; Mott, 1960; Mukhopadhyay, 1961; 1962; Choudhury and Basu, 1962; Mukherjee and Sarkar, 1963).

(2) Irredundant cover of the function is taken one at a time and then searched for whether different k-cells of the cover are incident in different $(k+1)$-cells. This can be done either comparing the algebraic expressions of the cells or by counting the total number of terms jointly covered by the cells. Depending on the nature of the function following different cases may arise in general.

Case (i).

All the k-cells that are present in the irredundant cover are such that each k-cell can combine with one or a group of k-cells to be incident in a distinct $(k+1)$-cell i.e., a k-cell does not share the parts of more than one $(k+1)$-cell. Method of finding three-level covers of such functions is illustrated by the following example.

Example :

Find the three-level cover of the five-variable switching function given by

$$f(x_4, x_3, x_2, x_1, x_0) = T = \Sigma\,(13, 14, 15, 16, 17, 18)$$

The basic cells of this function are

(1) (16, 17)

(2) (16, 18)

(3) (13, 15)

and (4) (14, 15)

All the basic cells being essential basic cells, the only two-level irredundant cover (or the minimal cover) of this function is

$$C = (16,\ 17)+(16,\ 18)+(13,\ 15)+(14,\ 15)$$

The cost of realisation of this two-level minimal cover is 20 diodes.

To find the three-level cover from the irredundant cover C, we note that the one-cells (16, 17) and (16, 18) are incident in a two-cell and one-cells (13, 15) and (14, 15) are incident in another two-cell. Neither of these one-cells (16, 17), (16, 18), (13, 15), (14, 15) is incident in any other two-cell. The three-level cover is given by

$$C_m = (16, 17, 18)+(13, 14, 15)$$

which requires 14 diodes for its mechanisation. This gives a saving of 6 diodes over the two-level minimal form.

In some cases we will find that the basic k-cells of the function may be incident in different $(k+1)$-cells but the irredundant covers of the function are such that in the covers each k-cell along with one or a group of k-cells is incident only in a distinct $(k+1)$-cell. Method of finding three-level covers of such functions is illustrated by the following example.

Example :

Find the three-level realisation of the following four variable switching function given by

$$f(x_3, x_2, x_1, x_0) = T = \Sigma(0, 1, 2, 5, 6, 7)$$

The basic cells of this function are:

(1)	(0, 1)
(2)	(0, 2)
(3)	(1, 5)
(4)	(2, 6)
(5)	(5, 7)
(6)	(6, 7)

The irredundant covers of the function are :

$$C_1 = (0, 1)+(0, 2)+(5, 7)+(6, 7)$$
$$C_2 = (0, 1)+(1, 5)+(2, 6)+(6, 7)$$
$$C_3 = (0, 2)+(1, 5)+(6, 7)$$
$$C_4 = (0, 1)+(2, 6)+(5, 7)$$

Of these irredundant covers, C_3 an C_4 represent the two-level minimal covers, cost of realisation of each of which is 12 diodes. Further, under the conditions we are considering (i.e., some of the k-cells of the cover being incident in $(k+1)$-cells, we do not get any three-level cover from these two minimal covers. But the irredundant covers C_1 and C_2 can be utilised to get three-level covers. In cover C_1, one-cells (0, 1) and (0, 2) are incident in a two-cell and one-cells (5, 7) and (6, 7) are incident in another two-cell, and these one-cells (0, 1), (0, 2), (5, 7) and (6, 7) are not incident in any other two-cells. The same is the case for cover C_2, where one-cells (0, 1) and (1, 5) are incident in a two-cell and one-cells (2, 6) and (6, 7) are incident in another two-cell. These three-level covers are

$$C_{1m} = (0, 1, 2)+(5, 6, 7)$$

$$C_{2m} = (0, 1, 5)+(2, 6, 7).$$

Cost of realisation of either of these three-level covers is 12 diodes which is the same as that required for the two-level minimal cover.

Case (ii).

In other cases, a k-cell may be incident in different $(k+1)$-cells along with different k-cells of the cover, that is, a k-cell will share parts of different

$(k+1)$-cells along with different k-cells. The method of finding three-level covers for such functions is illustrated by the following example.

Example :

Find the economical three-level realisations of the following six-variable switching function:

$$f(x_5, x_4, x_3, x_2, x_1, x_0) = T = \Sigma(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 24, 25, 26, 28, 30, 32, 33, 34, 36, 37, 38, 40, 41, 42, 44, 45, 48, 49, 50, 52, 53, 54, 56, 57, 58, 60)$$

The basic cells of this function along with their algebraic representations are

(1)	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15)	$-x_5'x_4'$
(2)	(0, 1, 4, 5, 8, 9, 12, 13, 32, 33, 36, 37, 40, 41, 44, 45)	$-x_4'x_1'$
(3)	(0, 1, 4, 5, 16, 17, 20, 21, 32, 33, 36, 37, 48, 49, 52, 53)	$-x_3'x_1'$
(4)	(0, 1, 8, 9, 16, 17, 24, 25, 32, 33, 40, 41, 48, 49, 56, 57)	$-x_2'x_1'$
(5)	(0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30)	$-x_5'x_0'$
(6)	(0, 2, 4, 6, 16, 18, 20, 22, 32, 34, 36, 38, 48, 50, 52, 54)	$-x_3'x_0'$
(7)	(0, 2, 8, 10, 16, 18, 24, 26, 32, 34, 40, 42, 48, 50, 56, 58)	$-x_2'x_0'$
(8)	(0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60)	$-x_1'x_0'$

All these basic cells being essential basic cells, only irredundant cover of the function is the disjunction of these essential basic cells.

To find the three-level covers from this two-level irredundant cover we shall search for k-cells of this cover incident in different $(k+1)$-cells. Here all the basic cells are four-cells and each of them is incident in different five-cells along with two or more of the remaining basic cells. If we designate the algebraic or numeric representation of the k-cells realised as being incident in $(k+1)$-cells by the term "three-level implicants", then the different three-level implicants that can be formed from the two-level basic cells of the irredundant cover of the given function will be

(i) $I = x_5'x_0'+x_5'x_4' = x'_5(x_0'+x_4')$

(ii) $J = x_5'x_4'+x'_4x'_1 = x'_4(x'_5+x'_1)$

(iii) $K = x_3'x_1'+x_3'x_0' = x'_3(x'_1+x'_0)$

(iv) $L = x'_2x'_1+x'_2x'_0 = x'_2(x'_1+x'_0)$

(v) $M = x'_4x'_1+x'_3x'_1+x'_2x'_1+x'_1x'_0 = x'_1(x'_4+x'_3+x'_2+x'_0)$

(vi) $N = x_5'x'_0+x'_3x'_0+x'_2x'_0+x'_1x'_0 = x'_0(x'_5+x'_3+x'_2+x'_1)$

such that in the covers each k-cell along with one or a group of k-cells is incident only in a distinct $(k+1)$-cell. Method of finding three-level covers of such functions is illustrated by the following example.

Example :

Find the three-level realisation of the following four variable switching function given by

$$f(x_3, x_2, x_1, x_0) = T = \Sigma(0, 1, 2, 5, 6, 7)$$

The basic cells of this function are:

(1)	(0, 1)
(2)	(0, 2)
(3)	(1, 5)
(4)	(2, 6)
(5)	(5, 7)
(6)	(6, 7)

The irredundant covers of the function are :

$$C_1 = (0, 1)+(0, 2)+(5, 7)+(6, 7)$$
$$C_2 = (0, 1)+(1, 5)+(2, 6)+(6, 7)$$
$$C_3 = (0, 2)+(1, 5)+(6, 7)$$
$$C_4 = (0, 1)+(2, 6)+(5, 7)$$

Of these irredundant covers, C_3 an C_4 represent the two-level minimal covers, cost of realisation of each of which is 12 diodes. Further, under the conditions we are considering (i.e., some of the k-cells of the cover being incident in $(k+1)$-cells, we do not get any three-level cover from these two minimal covers. But the irredundant covers C_1 and C_2 can be utilised to get three-level covers. In cover C_1, one-cells (0, 1) and (0, 2) are incident in a two-cell and one-cells (5, 7) and (6, 7) are incident in another two-cell, and these one-cells (0, 1), (0, 2), (5, 7) and (6, 7) are not incident in any other two-cells. The same is the case for cover C_2, where one-cells (0, 1) and (1, 5) are incident in a two-cell and one-cells (2, 6) and (6, 7) are incident in another two-cell. These three-level covers are

$$C_{1m} = (0, 1, 2)+(5, 6, 7)$$

$$C_{2m} = (0, 1, 5)+(2, 6, 7).$$

Cost of realisation of either of these three-level covers is 12 diodes which is the same as that required for the two-level minimal cover.

Case (ii).

In other cases, a k-cell may be incident in different $(k+1)$-cells along with different k-cells of the cover, that is, a k-cell will share parts of different

$(k+1)$-cells along with different k-cells. The method of finding three-level covers for such functions is illustrated by the following example.

Example :

Find the economical three-level realisations of the following six-variable switching function:

$$f(x_5, x_4, x_3, x_2, x_1, x_0) = T = \Sigma(0\ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 24, 25, 26, 28, 30, 32, 33, 34, 36, 37, 38, 40, 41, 42, 44, 45, 48, 49, 50, 52, 53, 54, 56, 57, 58, 60)$$

The basic cells of this function along with their algebraic representations are

(1)	(0, 1. 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. 13, 14, 15)	$-x_5'x_4'$
(2)	(0, 1, 4, 5, 8, 9, 12, 13, 32, 33, 36, 37, 40, 41, 44, 45)	$-x_4'x_1'$
(3)	(0, 1, 4, 5, 16, 17, 20, 21, 32, 33, 36, 37, 48, 49, 52, 53)	$-x_3'x_1'$
(4)	(0, 1, 8, 9, 16, 17, 24, 25, 32, 33, 40, 41, 48, 49, 56, 57)	$-x_2'x_1'$
(5)	(0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30)	$-x_5'x_0'$
(6)	(0, 2, 4, 6, 16, 18, 20, 22, 32, 34, 36, 38, 48, 50, 52, 54)	$-x_3'x_0'$
(7)	(0, 2, 8, 10, 16, 18, 24, 26, 32, 34, 40, 42, 48, 50, 56, 58)	$-x_2'x_0'$
(8)	(0, 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48, 52, 56, 60)	$-x_1'x_0'$

All these basic cells being essential basic cells, only irredundant cover of the function is the disjunction of these essential basic cells.

To find the three-level covers from this two-level irredundant cover we shall search for k-cells of this cover incident in different $(k+1)$-cells. Here all the basic cells are four-cells and each of them is incident in different five-cells along with two or more of the remaining basic cells. If we designate the algebraic or numeric representation of the k-cells realised as being incident in $(k+1)$-cells by the term "three-level implicants", then the different three-level implicants that can be formed from the two-level basic cells of the irredundant cover of the given function will be

(i) $I = x_5'x_0'+x_5'x_4' = x'_5(x_0'+x_4')$

(ii) $J = x_5'x_4'+x'_4x'_1 = x'_4(x'_5+x'_1)$

(iii) $K = x_3'x_1'+x_3'x_0' = x'_3(x'_1+x'_0)$

(iv) $L = x'_2x'_1+x'_2x'_0 = x'_2(x'_1+x'_0)$

(v) $M = x'_4x'_1+x'_3x'_1+x'_2x'_1+x'_1x'_0 = x'_1(x'_4+x'_3+x'_2+x'_0)$

(vi) $N = x_5'x'_0+x'_3x'_0+x'_2x'_0+x'_1x'_0 = x'_0(x'_5+x'_3+x'_2+x'_1)$

Next we shall find the irredundant three-level covers of the two-level basic cells by utilising a table (Table II).

TABLE II

	I	J	K	L	M	N
$x'_5x'_0$	1					1
$x'_5x'_4$	1	1				
$x'_4x'_1$		1			1	
$x'_3x'_1$			1		1	
$x'_3x'_0$			1			1
$x'_2x'_1$				1	1	
$x'_2x'_0$				1		1
$x'_1x'_0$					1	1

In this table, the column headings indicate three-level implicants whereas the row headings indicate two-level basic cells, 1's in the subcharts denoting which particular group of two-level basic cells form three-level implicants. From this table, the three-level irredundant covers (Mukherjee and Sarkar, 1963) are found to be :

(i) $IMKL$
(ii) IMN
(iii) JNM
(iv) $JNKL$

From each of these three-level irredundant covers different three-level covers are found by eliminating those cells which occur more than once, thus introducing redundancy. Let us first take up the three-level irredundant cover $IMKL$. In this three-level irredundant cover, two-level basic cell $x'_1x'_3$ occurs in both of the three-level implicants M and K. Similarly, two-level basic cell $x'_1x'_2$ occurs in both of M and L. Since in the minimal three-level cover the same two-level basic cell cannot occur more than once, so to avoid redundancy, the three-level irredundant cover $IMKL$ is expanded in the form of a table (Table III) as shown below. Here $x'_1x'_3$ can occur either in M or K and with each occurrence of $(x'_1x'_3)$ there are two possible choices of $(x'_1x'_2)$ in either M or L. Hence altogether we get four possible configurations, giving four possible three-level covers.

TABLE III

I			M					K			L		
$x'_5(x'_0+x'_4)$			$x'_1(x'_4+x'_3+x'_2+x'_0)$					$x'_3(x'_1+x'_0)$			$x'_2(x'_1+x'_0)$		
1	1	1	1	1	0	0	1	1	1	1	1	1	1
1	1	1	1	1	0	1	1	1	1	1	1	0	1
1	1	1	1	1	1	0	1	1	0	1	1	1	1
1	1	1	1	1	1	1	1	1	0	1	1	0	1

From this, the three-level covers are :

$$
\begin{aligned}
C_{im_1} &= x'_5(x'_0+x'_4)+x'_1(x'_4+x'_0)+x_3'(x'_1+x'_0)+x'_2(x'_1+x'_0) \\
C_{im_2} &= x'_5(x'_0+x'_4)+x_1'(x'_4+x'_2+x'_0)+x'_3(x'_1+x'_0)+x'_2x'_0 \\
C_{im_3} &= x'_5(x'_0+x'_4)+x_1'(x'_4+x'_3+x'_0)+x'_3\,x'_0+x'_2(x'_1+x'_0) \\
C_{im_4} &= x'_5(x'_0+x'_4)+x'_1(x'_4+x'_3+x'_2+x'_0)+x'_3\,x'_0+x'_2\,x'_0
\end{aligned}
$$

Similarly, expanding the cover *IMN*, we get

TABLE IV

I			M					N				
$x'_5(x'_0+x'_4)$			$x'_1(x'_4+x'_3+x'_2+x'_0)$					$x'_0(x'_5+x'_3+x'_2+x'_1)$				
1	1	1	1	1	1	1	1	1	0	1	1	0
1	1	1	1	1	1	1	0	1	0	1	1	1
1	0	1	1	1	1	1	1	1	1	1	1	0
1	0	1	1	1	1	1	0	1	1	1	1	1

From this, the simplest three-level covers are

$$
\begin{aligned}
C_{jm_1} &= x'_5(x'_0+x'_4)+x'_1(x'_4+x'_3+x'_2+x'_0)+x'_0(x'_3+x'_2) \\
C_{jm_2} &= x'_5(x'_0+x'_4)+x'_1(x_4'+x'_3+x'_2)+x'_0(x'_3+x'_2+x'_1) \\
C_{jm_3} &= x'_5x'_4+x'_1(x'_4+x'_3+x'_2+x'_0)+x'_0(x'_5+x'_3+x'_2) \\
C_{jm_4} &= x'_5x'_4+x'_1(x'_4+x'_3+x'_2)+x'_0(x'_5+x'_3+x'_2+x'_1)
\end{aligned}
$$

Likewise, the three-level covers from the irredundant covers *JNM* and *JNKL* are, respectively :

$$
\begin{aligned}
C_{km_1} &= x'_4(x'_5+x'_1)+x'_0(x'_5+x'_3+x'_2+x'_1)+x'_1(x'_3+x'_2) \\
C_{km_2} &= x'_4(x'_5+x'_1)+x'_0(x'_5+x'_3+x'_2)+x_1'(x'_3+x'_2+x'_0) \\
C_{km_3} &= x'_4x'_5+x'_0(x'_5+x'_3+x'_2+x'_1)+x'_1(x'_4+x'_3+x'_2) \\
C_{km_4} &= x'_4x'_5+x'_0(x'_5+x'_3+x'_2)+x'_1(x'_4+x'_3+x'_2+x'_0)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{lm_1} &= x'_4(x'_5+x'_1)+x_0'(x'_5+x'_1)+x'_3(x'_1+x'_0)+x'_2(x'_1+x'_0) \\
C_{lm_2} &= x'_4(x'_5+x'_1)+x'_0(x'_5+x'_2+x'_1)+x_3'(x'_1+x'_0)+x'_2x'_1 \\
C_{lm_3} &= x'_4(x'_5+x'_1)+x'_0(x'_5+x'_3+x'_1)+x_3'x'_1+x'_2(x'_1+x'_0) \\
C_{lm_4} &= x'_4(x'_5+x'_1)+x'_0(x'_5+x'_3+x'_2+x'_1)+x'_3x'_1+x'_2x'_1
\end{aligned}
$$

From an inspection of all these three-level covers, we see that the most economical three-level cover is

$$
\begin{aligned}
C_{min} &= x'_5(x'_0+x'_4)+x'_1(x'_4+x'_0)+x'_3(x'_1+x'_0)+x'_2(x'_1+x'_0) \\
&= (x'_4+x'_0)(x'_5+x'_1)+(x'_1+x'_0)(x'_3+x'_2)
\end{aligned}
$$

The three-level circuits of the switching functions, whose irredundant covers can be written in a compact form by following a procedure recently suggested by Mukherjee and Sarkar (1963), can be found readily even when the total number of such covers are very large.

These compact forms of representation aid in obtaining the three-level expressions of switching function without trying all the irredundant covers one at a time, which involves large amount of labour when the total number of such covers is very large. The following example will suggest the method of obtaining three-level expressions of switching function starting from its compact forms of irredundant expressions.

Example :

Find the economical three-level realisation of the seven-variable switching function given by

$$f(x_6, x_5, x_4, x_3, x_2, x_1, x_0) = T = \Sigma(0, 2, 5, 8, 10, 12, 13, 16, 18, 21, 24, 26, 28, 29, 30, 32, 34, 37, 39, 40, 42, 45, 46, 48, 50, 53, 55, 56, 58, 61, 64, 65, 67, 72, 73, 77, 78, 79, 80, 81, 83, 88, 89, 91, 95, 100, 102, 106, 107, 111, 114, 116, 120, 122, 126, 127)$$

The basic cells of this function, along with their algebraic representations, are :

(1) (0, 2, 8, 10, 16, 18, 24, 26, 32, 34, 40, 42, 48, 50, 56 58)* $= x'_6x'_2x'_0$ *A**

(2) (5, 13, 21, 29, 37, 45, 53, 61)* $= x'_6\, x_2x'_1x_0$ *B**

(3) (64, 65, 72, 73, 80, 81, 88, 89) $= x'_6\, x'_5x'_2x'_1$ *C*

(4) (0, 8, 16, 24, 64, 72, 80, 88) $= x_5x'_2x'_1x'_0$ *D*

(5) (79, 95, 111, 127) $= x_6x_3x_2x_1x_0$ *E*

(6) (81, 83, 89, 91) $= x_6x'_5x_4x'_2x_0$ *F*

(7) (56, 58, 120, 122) $= x_5x_4x_3x'_2x'_0$ *G*

(8) (50, 58, 114, 122)* $= x_5x_4x'_2x_1x'_0$ *H**

(9) (42, 58, 106, 122) $= x_5x_3x'_2x_1x'_0$ *I*

(10) (37, 39, 53, 55)* $= x'_6x_5x'_3x_2x_0$ *J**

(11) (65, 67, 81, 83)* $= x_6x'_5x'_3x'_2x_0$ *K**

(12) (24, 56, 88, 120) $= x_4x_3x'_2x'_1x_0'$ *L*

(13) (24, 26, 28, 30)* $= x'_6x'_5x_4x_3x'_0$ *M**

(14) (12, 13, 28, 29) $= x'_6x'_5x_3x_2x'_1$ *N*

(15) $(8, 12, 24, 28) = x'_6x'_5x_3x'_1x'_0$ $\quad O$

(16) $(126, 127) = x_6x_5x_4x_3x_2x_1$ $\quad P$

(17) $(122, 126) = x_6x_5x_4x_3x_1x'_0$ $\quad Q$

(18) $(107, 111) = x_6x_5x'_4x_3x_1x_0$ $\quad R$

(19) $(91, 95) = x_6x'_5x_4x_3x_1x_0$ $\quad S$

(20) $(106, 107) = x_6x_5x'_4x_3x'_2x_1$ $\quad T$

(21) $(78, 79)^* = x_6x'_5x'_4x_3x_2x_1$ $\quad U^*$

(22) $(77, 79) = x_6x'_5x'_4x_3x_2x_0$ $\quad V$

(23) $(100, 116)^* = x_6x_5x'_3x_2x'_1x'_0$ $\quad W^*$

(24) $(100, 102)^* = x_6x_5x'_4x'_3x_2x'_0$ $\quad X^*$

(25) $(73, 77) = x_6x'_5x'_4x_3x'_1x_0$ $\quad Y$

(26) $(42, 46)^* = x'_6x_5x'_4x_3x_1x'_0$ $\quad Z^*$

(27) $(13, 77) = x'_5x'_4x_3x_2x'_1x_0$

In the above, basic cells marked by asterisk are essential basic cells. Therefore these basic cells will occur in all the irredundant covers of the function.

The compact expressions for the irredundant forms of this function are given below.

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} C \begin{bmatrix} V \\ Y \\ a \end{bmatrix} \begin{bmatrix} G \\ L \end{bmatrix} FEIR \begin{bmatrix} P \\ Q \end{bmatrix} \qquad \ldots \quad \text{(i)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} C \begin{bmatrix} V \\ Y \\ a \end{bmatrix} \begin{bmatrix} G \\ L \end{bmatrix} FET \begin{bmatrix} P \\ Q \end{bmatrix} \qquad \ldots \quad \text{(ii)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} C \begin{bmatrix} V \\ Y \\ a \end{bmatrix} \begin{bmatrix} G \\ L \end{bmatrix} SIR \begin{bmatrix} P \\ QE \end{bmatrix} \qquad \ldots \quad \text{(iii)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} C \begin{bmatrix} V \\ Y \\ a \end{bmatrix} \begin{bmatrix} G \\ L \end{bmatrix} STE \begin{bmatrix} P \\ Q \end{bmatrix} \qquad \ldots \quad \text{(iv)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} C \begin{bmatrix} V \\ Y \\ a \end{bmatrix} \begin{bmatrix} G \\ L \end{bmatrix} STRP \qquad \ldots \quad \text{(v)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} DYF \begin{bmatrix} G \\ L \end{bmatrix} IRE \begin{bmatrix} P \\ Q \end{bmatrix} \qquad \ldots \quad \text{(vi)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} DYF \begin{bmatrix} G \\ L \end{bmatrix} IRSP \quad \ldots \text{(vii)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} DYF \begin{bmatrix} G \\ L \end{bmatrix} TE \quad \ldots \text{(viii)}$$

$$(ABHJKMUWXZ) \begin{bmatrix} O \\ N \end{bmatrix} DYF \quad TSRP \quad \text{(ix)}$$

In the expressions for the irredundant forms of the function it will be found that certain prime implicants are written in a column matrix. In order to obtain all the irredundant covers, each of these expressions has to be expanded. Of the prime implicants written in any column matrix, only one of the prime implicants taken at a time can be associated with other prime implicants of the expression. In the expression (i), the prime implicants $A, B, H, J, K, M, U, W, X, Z, C, F, E, I, R,$ have to be associated with either O or N (but not with ON jointly). With these two different choices must be associated three different choices of the prime implicants from the column matrix $\begin{bmatrix} V \\ Y \\ a \end{bmatrix}$, two different choices from the column matrix $\begin{bmatrix} G \\ L \end{bmatrix}$ and two different choices from the column matrix $\begin{bmatrix} P \\ Q \end{bmatrix}$. Thus a total of $2 \times 3 \times 2 \times 2 = 24$ different irredundant covers of the function will be obtained from the expression (i).

To find three-level covers from these compact forms of irredundant covers, we shall first form the "three-level implicants" from the sets of prime implicants. These three-level implicants are :

(1) $CD = x'_5 x'_2 x'_1 (x_6 + x'_0)$

(2) $GH = x_5 x_4 x'_2 x'_0 (x_3 + x_1)$

(3) $GI = x_5 x_3 x'_2 x'_0 (x_4 + x_1)$

(4) $GL = x_4 x_3 x'_2 x'_0 (x_5 + x'_1)$

(5) $FK = x_6 x'_5 x'_2 x_0 (x_4 + x'_3)$

(6) $MO = x'_6 x'_5 x_3 x'_0 (x_4 + x'_1)$

(7) $HI = x_5 x'_2 x_1 x'_0 (x_4 + x_3)$

(8) $NO = x'_6 x'_5 x_3 x'_1 (x_2 + x'_0)$

(9) $PQ = x_6 x_5 x_4 x_3 x_1 (x_2 + x'_0)$

(10) $RT = x_6 x_5 x'_4 x_3 x_1 (x'_2 + x_0)$

(11) $UV = x_6 x'_5 x'_4 x_3 x_2 (x_1 + x_0)$

$$\begin{array}{lll} (12) & VY & = x_6x'_5x'_4x_3x_0(x_2+x'_1) \\ (13) & WX & = x_6x_5x_3'x_2x'_0(x_4'+x'_1) \\ (14) & Ya & = x'_5x'_4x_3x'_1x_0(x_6+x_2) \end{array}$$

The simplified three-level forms are found out from the compact two-level irredundant forms in the following way. We have already observed that the irredundant cover

$$ABHJKMUWXZ \begin{bmatrix} O \\ N \end{bmatrix} C \begin{bmatrix} V \\ Y \\ a \end{bmatrix} \begin{bmatrix} G \\ L \end{bmatrix} FEIR \begin{bmatrix} P \\ Q \end{bmatrix} \quad \ldots \quad \text{(i)}$$

when expanded will yield 24 different irredundant covers of the given function. Some prime implicants will be common in all the covers and the different covers are obtained from the combination of the prime implicants in the column matrix. Therefore for obtaining economic three-level circuits from the above expression (i), the expression is simplified by retaining only those prime implicants in the column matrix which can combine with others to form three-level implicants. Following this procedure, the column matrix $\begin{bmatrix} O \\ N \end{bmatrix}$ is simplified by retaining only the prime implicant O. Because from the list of the three-level implicants we find that the prime implicant N can combine with O only and in any irredundant cover O and N cannot occur jointly. Similarly, column matrix $\begin{bmatrix} V \\ Y \\ a \end{bmatrix}$ is simplified by retaining only V.

Hence the expression (i) can be written as,

$$ABHJKMUWXZOCV \begin{bmatrix} G \\ L \end{bmatrix} FEIR \begin{bmatrix} P \\ Q \end{bmatrix} \quad \ldots \quad \text{(x)}$$

Thus without checking all the 24 covers, by simple inspection, we have found out that out of the 24 irredundant covers only four need be tried for economic three-level circuit synthesis.

Now noting that W combines with X, M with O, F with K, V with U, G with either of H or I, each to give three-level implicants, the simplified three-level expression that can be written from the compact two-level irredundant cover (i) is

$$ABJZCER(WX)(MO)(FK)(VU) \begin{bmatrix} P \\ Q \end{bmatrix} \begin{bmatrix} (GH) & I \\ (GI) & H \\ (HI) & \begin{bmatrix} G \\ L \end{bmatrix} \end{bmatrix} \quad \ldots \quad \text{(xi)}$$

where the expressions within the first brackets are three-level implicants. Similar three-level expressions can be obtained from other compact two-level irredundant covers. Total number of three-level covers that can be obtained by expanding the expression (xi) is $2 \times 4 = 8$.

All the three-level covers that can be obtained from the irredundant forms (i-ix) are given below, along with their cost numbers.

Three-level covers	*Cost number*
(i) $ABJZCER(WX)(MO)FK)(VU) \begin{bmatrix} P \\ Q \end{bmatrix} \begin{bmatrix} (GH) & I \\ (GI) & H \\ (HI) & \begin{bmatrix} G \\ L \end{bmatrix} \end{bmatrix}$	... 95
(ii) $ABJZCET(WX)(MO)(FK)(VU) \begin{bmatrix} P \\ Q \end{bmatrix} (GH)$	... 89
(iii) $ABJKZCSR(WX)(MO)(VU)P \begin{bmatrix} (GH) & I \\ (GI) & H \\ (HI) & \begin{bmatrix} G \\ L \end{bmatrix} \end{bmatrix}$	... 94
(iv) $ABJKZCSTE \begin{bmatrix} P \\ Q \end{bmatrix} (WX)(MO)(VU)(GH)$	... 94
(v) $ABJKZCSP(WX)(MO)(VU)(GH)(RT)$	... 90
(vi) $ABJUZDYRE \begin{bmatrix} P \\ Q \end{bmatrix} (WX)(MO)(FK) \begin{bmatrix} (GH) & I \\ (GI) & H \\ (HI) & \begin{bmatrix} G \\ L \end{bmatrix} \end{bmatrix}$	... 100
(vii) $ABJUZDYRSP(WX)(MO)(FK) \begin{bmatrix} (GH) & I \\ (GI) & H \\ (HI) & \begin{bmatrix} G \\ L \end{bmatrix} \end{bmatrix}$	... 101
(viii) $ABJUZDYTE \begin{bmatrix} P \\ Q \end{bmatrix} (WX)(MO)(FK)(GH)$	... 94
(ix) $ABJUZDYSP(WX)(MO)(FK)(GH)(RT)$	... 97

From the above three-level covers, we see that the most economical three-level cover is

$$
\begin{aligned}
C_m &= ABJEZCTP(WX)(MO)(VU)(GH)(FK) \\
&= x'_6 x'_2 x'_0 + x'_6 x_2 x'_1 x_0 + x_6 x'_5 x'_2 x'_1 + x'_6 x_5 x'_3 x_2 x_0 + x_6 x_3 x_2 x_1 x_0 \\
&\quad + x_6 x_5 x_4 x_3 x_2 x_1 + x_6 x_5 x'_4 x_3 x'_2 x_1 + x'_6 x_5 x'_4 x_3 x_1 x'_0 + x'_6 x'_5 x_3 x'_0 (x_4 + x'_1) \\
&\quad + x_6 x'_5 x'_4 x_3 x_2 (x_1 + x_0) + x_6 x'_5 x'_2 x_0 (x_4 + x'_3) + x_6 x_5 x'_3 x_2 x'_0 (x'_4 + x'_1) \\
&\quad + x_5 x_4 x'_2 x'_0 (x_3 + x_1)
\end{aligned}
$$

CONCLUSION

A simple and straightforward method is suggested for obtaining economic three-level circuit realisations of switching functions starting from the two-level irredundant covers. When the k-cells of the functions share the parts of different $(K+1)$-cells, it is possible to find the best three-level cover of the function by finding the three-level irredundant covers of the prime implicants which can form different three-level implicants. In cases where the function has a large number of two-level irredundant covers, if these covers can be written in compact forms, then the three-level covers of the function can be found very easily from the two-level irredundant covers without trying all of them at a time.

ACKNOWLEDGMENT

The authors wish to express their indebtedness to Prof. J. N. Bhar, D.Sc., F.N.I., for guidance and keen interest in the work. One of the authors, Shri Das also thankfully acknowledges the award of a Research Fellowship by the U.G.C., Government of India, while doing the work (May–August 1962).

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